Spectral properties of the Preisach hysteresis model with random input. II. Universality classes for symmetric elementary loops

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The Preisach model with symmetric elementary hysteresis loops and uncorrelated input is treated analytically in detail. It is shown that the appearance of long-time tails in the output correlations is a quite general feature of this model. The exponent η of the algebraic decay $t^{-\eta}$, which may take any positive value, is determined by the tails of the input and the Preisach density. We identify the system classes leading to identical algebraic tails. These results imply the occurrence of $1/f$ noise for a large class of hysteretic systems.

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I. INTRODUCTION

In a previous paper (Part I, [1]) we derived general expressions for the spectral density of the output of Preisach transducers for uncorrelated input. In an explicit example we were able to show the occurrence of a power-law decay of the corresponding autocorrelation function. The output of the Preisach model can be considered as a superposition of the output of infinitely many elementary hysteresis loops [2]. In the general model these loops are those of nonideal relays with arbitrary switching values α and β. In many physical situations, however, the elementary loops can be assumed to be symmetric, which for the relay loops of the Preisach model means $\beta=-\alpha$. Apart from its general importance this case is interesting because it allows for a rather complete understanding of the output properties for uncorrelated input. This symmetric model has been considered previously for input generated by an Ornstein-Uhlenbeck process [3]. In this case, however, explicit analytic results for the spectral density of the output cannot be given. In contrast, for uncorrelated input we are able to provide exact results for the spectral density and for the decay of the autocorrelation function. Since this case is considerably simpler, we are able to provide here a rather complete picture of the mechanisms for the long-time tails. Especially the origin and the range of possible exponents of the algebraic decay will be evaluated systematically and the appearance of $1/f$ noise for a wide range of systems will be shown analytically.

II. DEFINITIONS

We provide here briefly the definitions needed for the presentation of our results below. More details can be found in Part I [1]. The Preisach model is characterized by the so-called Preisach operator $\mathcal{P}$, which acts on an input time series $x(t)$ to produce the output $y(t)$. In the symmetric case the input-output relation can be written as

$$y(t)=\mathcal{P}[x(t)]=\int d\alpha\mu(\alpha)s_{\alpha,-\alpha}[x(t)], \quad (1)$$

where $s_{\alpha,-\alpha}[x(t)]\in\{-1,+1\}$ is the output of a symmetric nonideal relay with initial state $s_{\alpha\beta}(t_0)=s_0$ for a given input time series $x(t)$, $t\geq t_0$. It is characterized by a symmetric, rectangular elementary hysteresis loop, for which the output can be written as

$$s_{\alpha,-\alpha}[x(t)]=\begin{cases}+1 \text{ if there exists } t_1\in[t_0,t] \text{ such that } x(t_1)\geq\alpha \text{ and } x(\tau)>-\alpha \quad \text{for all } \tau\in[t_1,t]\\ -1 \text{ if there exists } t_1\in[t_0,t] \text{ such that } x(t_1)\leq-\alpha \text{ and } x(\tau)<\alpha \quad \text{for all } \tau\in[t_1,t]\\ s_0\in\{-1,+1\} \text{ if } -\alpha<x(\tau)<\alpha \quad \text{for all } \tau\in[t_0,t]\end{cases}. \quad (2)$$

We consider input and output sequences in discrete time $t=0,1,2,\ldots$, with elements $\{x(t)\}$ and $\{y(t)\}$, respectively. The results presented below are all obtained for $\{x(t)\}$ being a stochastic process consisting of independent identically distributed (i.i.d.) random variables with density $\rho(x)$. We consider two-point correlation functions of the output in the stationary case given by

$$C(\tau)=\langle y(0)y(\tau)\rangle-\langle y\rangle^2, \quad (3)$$

respectively, its Z transform

$$\tilde{C}(z)=\sum_{\tau=0}^{\infty}C(\tau)z^{-\tau}. \quad (4)$$

From the latter the power spectral density of the output

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$$S(\omega) = \sum_{\tau=-\infty}^{\infty} C(\tau)e^{i\omega\tau} = \lim_{N\to\infty}\left\langle \frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{i\omega t}\right|^2\right\rangle, \quad (5)$$

with $-\pi < \omega \leq \pi$, can be obtained as

$$S(\omega) = 2\,\mathrm{Re}[\tilde{C}(z=e^{i\omega})] - C(0), \quad (6)$$

with $C(0)=\lim_{z\to\infty}\tilde{C}(z)$.

III. RESULTS

A. General results

In our previous paper (Part I, [1]) we derived for uncorrelated input the following general expression for the Z transform of the output of a symmetric Preisach model:

$$\tilde{C}(z) = \int_0^\infty d\alpha\mu(\alpha)\frac{1}{1-F(-\alpha,\alpha)}\int_\alpha^\infty d\alpha'\mu(\alpha') \times \frac{F(-\infty,-\alpha')F(\alpha',\infty)}{1-F(-\alpha',\alpha')}\frac{4z}{z-F(-\alpha',\alpha')} + \int_0^\infty d\alpha\mu(\alpha)\frac{F(-\infty,-\alpha)F(\alpha,\infty)}{1-F(-\alpha,\alpha)}\int_0^\alpha d\alpha'\mu(\alpha') \times \frac{1}{1-F(-\alpha',\alpha')}\frac{4z}{z-F(-\alpha',\alpha')}, \quad (7)$$

where $F(a,b)$ is related to the density of the input $\rho(x)$ by $F(a,b)=\int_a^b\rho(x)dx$ and $\mu(\alpha)$ is the Preisach density of the symmetric elementary hysteresis loops. We now assume in addition that the input distribution is symmetric $\rho(x)=\rho(-x)$. Then by exploiting this symmetry for the integrated density F, i.e., $F(-\infty,-\alpha)=F(\alpha,\infty)$ and $1-F(-\alpha,\alpha)=2F(\alpha,\infty)$, the expression in Eq. (7) further simplifies to

$$\tilde{C}(z) = \int_0^\infty d\alpha\mu(\alpha)\int_\alpha^\infty d\alpha'\mu(\alpha')\frac{F(\alpha',\infty)}{F(\alpha,\infty)}\frac{z}{z-1+2F(\alpha',\infty)} + \int_0^\infty d\alpha\mu(\alpha)\int_0^\alpha d\alpha'\mu(\alpha')\frac{F(\alpha,\infty)}{F(\alpha',\infty)}\frac{z}{z-1+2F(\alpha',\infty)}. \quad (8)$$

This expression suggests a further transformation. We define new variables $u\equiv 2F(\alpha,\infty)$, which as function of α decreases monotonically with increasing α from $u(\alpha=0)=1$ to $u(\alpha=\infty)=0$, and analogously $v\equiv 2F(\alpha',\infty)$. Substituting these into the integrals of Eq. (8), one obtains

$$\tilde{C}(z) = \int_0^1 du\tilde{\mu}(u)\int_0^u dv\tilde{\mu}(v)\frac{v}{u}\frac{z}{z-1+v} + \int_0^1 du\tilde{\mu}(u)\int_u^1 dv\tilde{\mu}(v)\frac{u}{v}\frac{z}{z-1+v}, \quad (9)$$

where the expression

$$\tilde{\mu}(u) \equiv \frac{\mu(\alpha(u))}{2\rho(\alpha(u))} \quad (10)$$

is an effective Preisach density, with $\alpha(u)$ the inverse function of $u(\alpha)=2F(\alpha,\infty)$. One easily verifies that $\tilde{\mu}(u)$ is properly normalized $\int_0^1 du\tilde{\mu}(u)=1$, and that a Preisach density equal to $\mu(\alpha)$ for $0\leq\alpha\leq 1$ and zero elsewhere combined with a constant input density $\rho(x)=\frac{1}{2}$ for $|x|\leq 1$ and zero elsewhere, yields $\tilde{\mu}(u)=\mu(1-u)$. This is interesting because first this means that there are classes of systems, i.e., combinations of input and Preisach densities, which yield the same effective Preisach density $\tilde{\mu}(u)$ and therefore also the same spectral density of the output. This fact will be exploited in the following sections. Second, any pair of input density $\rho(x)$ and Preisach density $\mu(\alpha)$ is equivalent to one with constant input density, thus justifying the notion of an effective density for the resulting $\tilde{\mu}(u)$. Especially all systems where the Preisach density $\mu(\alpha)$ and twice the input density $\rho(x)$ have the same functional dependence, e.g., both are Gaussian distributed with the same variance, are equivalent to the case $\tilde{\mu}(u)=1$. The latter can be regarded as the representative of the simplest case with Preisach density $\mu(\alpha)=1$ for $0\leq\alpha\leq 1$ and zero elsewhere, and input density $\rho(x)=\frac{1}{2}$ for $|x|\leq 1$ and zero elsewhere.

B. Constant Preisach and constant input density

This will also be the first special case for which we can evaluate the integrals of Eq. (8), or equivalently, Eq. (9) with $\tilde{\mu}(u)=1$. One obtains the exact result

$$\tilde{C}(z) = \frac{3}{2}z + \frac{z}{2}(z-1)\ln\left(\frac{z-1}{z}\right) + z(z-1)\mathrm{Li}_2\left(\frac{1}{1-z}\right), \quad (11)$$

where $\mathrm{Li}_2(z)=\sum_{n=1}^\infty z^n/n^2=-\int_0^z dt\frac{1}{t}\ln(1-t)$ is the Euler dilogarithm. Note that in this case $C(t=0)=\lim_{z\to\infty}\tilde{C}(z)=\frac{1}{2}$. Since the long-time behavior of $C(t)$ is determined by the behavior of $\tilde{C}(z)$ near $z=1$, we need the corresponding asymptotic expansion. This is obtained by inserting the transformation [4] $\mathrm{Li}_2(\frac{1}{1-z})=-\frac{\pi^2}{6}-\frac{1}{2}\ln^2(z-1)-\mathrm{Li}_2(1-z)$ into Eq. (11), which results in

$$\tilde{C}(z) \sim \frac{3}{2} + \frac{1}{2}\left[3-\frac{\pi^2}{3}+\ln(z-1)-\ln^2(z-1)\right](z-1) + O((z-1)^2). \quad (12)$$

This shows that the first derivative $\tilde{C}^{(1)}(z)$ of $\tilde{C}(z)$ is logarithmically divergent near $z=1$. Indeed one has

$$\tilde{C}^{(1)}(z) \sim -\frac{1}{2}\ln^2(z-1) - \frac{1}{2}\ln(z-1) + 2 - \frac{\pi^2}{6} + O[(z-1)]. \quad (13)$$

Applying Karamata's Tauberian theorem for power series [7], analogous to its application in Part I [1], to the leading term $\tilde{C}^{(1)}(z)\sim-\frac{1}{2}\ln^2(z-1)$, yields for the asymptotic behavior of $C(t)$,

$$C(t) \sim t^{-2}\ln t. \quad (14)$$

The corresponding behavior in the spectral density at small frequencies is obtained by expanding $S(\omega)=\tilde{C}(e^{i\omega})+\tilde{C}(e^{-i\omega})-\frac{1}{2}$ with $\tilde{C}$ from Eq. (11) around $\omega=0$. One obtains

$$S(\omega) \sim \frac{5}{2} + \pi\omega \ln \omega - \frac{\pi}{2}\omega + O(\omega^2). \tag{15}$$

Since the Z transform of an exponentially decaying function $C_0(t)=b^t=\exp(-\lambda t)$ with $\lambda=|\ln b|$, $0<b<1$, is given by $\tilde{C}_0(z)=\frac{z}{z-b}$, we see that Eq. (8) can be regarded as the superposition of infinitely many exponentially decaying contributions with decay rates $\lambda(\alpha)=|\ln[1-2F(\alpha,\infty)]|$. The smallest decay rates, responsible for any long-time tails in $C(t)$, come from α values near the maximal value $\alpha_{\max}$ (which may be infinity) since there $\lambda(\alpha)\sim 2\,F(\alpha,\infty)$, which becomes zero as α approaches $\alpha_{\max}$. This argument holds if the range of possible positive input values [symmetry of $\rho(x)$] coincides with, or is contained in the range of possible threshold values. This shows that the way $F(\alpha,\infty)=\int_\alpha^\infty \rho(x)dx$ behaves for $\alpha\to\alpha_{\max}$ determines the form of the long-time tails of the autocorrelation function. To investigate this effect we consider a whole family of input distributions to a Preisach model with constant Preisach density.

C. Constant Preisach and power-law input density

To be specific we consider now symmetric input densities given by $\rho(x)=\frac{\nu}{2}(1-|x|)^{\nu-1}$ for $|x|\le 1$, and zero elsewhere. The parameter ν can take any positive value $\nu>0$. The Preisach density μ is given by $\mu(\alpha)=1$ for $0\le\alpha\le 1$ and zero elsewhere, so $\alpha_{\max}=1$. For this case $u(\alpha)=2F(\alpha,1)=(1-\alpha)^\nu$ and thus $\alpha(u)=1-u^{1/\nu}$. From this we get an effective Preisach density, Eq. (10), which reads

$$\tilde{\mu}(u) = \frac{1}{\nu}u^{-1+1/\nu}. \tag{16}$$

For this effective density the integrals in Eq. (9) can be evaluated exactly. One obtains for $\nu\neq 1$ the following expression:

$$\tilde{C}(z) = \frac{z}{z-1}\frac{1}{\nu^2-1}\left[\nu F\left(1,\beta_1;\beta_1+1;\frac{1}{1-z}\right) - F\left(1,\beta_2;\beta_2+1;\frac{1}{1-z}\right)\right], \tag{17}$$

where $F(a,b;c;z)$ is the hypergeometric function $F(a,b;c;z)\equiv\sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$, with $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$ denoting the Pochhammer symbol. The parameters β_1 and β_2 are related to the exponent ν by $\beta_1=2/\nu$ and $\beta_2=1+1/\nu$, respectively. The value for $C(t=0)$ needed for the evaluation of the spectrum is given by $\lim_{z\to\infty}\tilde{C}(z)=\frac{1}{1+\nu}$. As before we have to study the behavior of the expression Eq. (17) near $z=1$ in order to obtain the long-time behavior of the autocorrelation function. For this purpose we apply a relation connecting the hypergeometric functions of reciprocal arguments [Eq. (15.3.7) in [5]] to $F(1,\beta;\beta+1;\frac{1}{1-z})$. For noninteger values of β one finds

$$F\left(1,\beta;\beta+1;\frac{1}{1-z}\right) = \Gamma(1+\beta)\Gamma(1-\beta)(z-1)^\beta + \frac{\Gamma(\beta+1)\Gamma(\beta-1)}{\Gamma(\beta)^2}(z-1) \times F(1,1-\beta;2-\beta;1-z). \tag{18}$$

Using the properties of the gamma function one obtains with the series expansion of $F(1,1-\beta;2-\beta;1-z)$ the following identity:

$$\frac{1}{z-1}F\left(1,\beta;\beta+1;\frac{1}{1-z}\right) = \frac{\beta\pi}{\sin\beta\pi}(z-1)^{\beta-1} - \beta\sum_{k=0}^\infty \frac{(1-z)^k}{k+1-\beta}. \tag{19}$$

From the right-hand side we see that for a given β the lowest derivative of this expression, which becomes singular for $z\to 1$, is the nth derivative, where $n=\text{int}(\beta)$ is the largest integer smaller than β. This means that for $\nu>1$, i.e., $\beta_1<\beta_2$, in Eq. (17) only the term $\tilde{C}_1(z)=\frac{\nu}{\nu^2-1}\frac{z}{z-1}F(1,\beta_1;\beta_1+1;\frac{1}{1-z})$ has to be considered, whereas for $0<\nu<1$ the term $\tilde{C}_2(z)=\frac{1}{1-\nu^2}\frac{z}{z-1}F(1,\beta_2;\beta_2+1;\frac{1}{1-z})$ determines the singular behavior of $\tilde{C}(z)$ or its derivatives as $z\to 1$. Denoting by $\tilde{C}^{(n)}(z)$, the nth derivative of $\tilde{C}(z)$, and similarly for $\tilde{C}_1(z)$ and $\tilde{C}_2(z)$, one finds the following ranges and corresponding singular behavior for $z\to 1$:

$$2<\nu<\infty,\ \ 0<\beta_1<1:$$

$$\tilde{C}(z) \sim \tilde{C}_1(z) \sim \frac{\nu}{\nu^2-1}\Gamma(1+\beta_1)\Gamma(1-\beta_1)(z-1)^{\beta_1-1}, \tag{20}$$

$$1<\nu<2,\ \ 1<\beta_1<2:$$

$$\tilde{C}_{(1)}(z) \sim \tilde{C}_1^{(1)}(z) \sim \frac{\nu}{\nu^2-1}\Gamma(1+\beta_1)\Gamma(1-\beta_1) \times(\beta_1-1)(z-1)^{\beta_1-2}, \tag{21}$$

$$\frac{1}{n}<\nu<\frac{1}{n-1},\ \ n<\beta_2<n+1,\ \ n=2,3,\ldots:$$

$$\tilde{C}^{(n)}(z) \sim \tilde{C}_2^{(n)}(z) \sim \frac{1}{1-\nu^2}\Gamma(1+\beta_2)\Gamma(1-\beta_2) \times\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-n)}(z-1)^{\beta_2-n-1}, \tag{22}$$

where for clarity the abbreviations $\beta_1=2/\nu$ and $\beta_2=1+1/\nu$ have been used again. Applying again Karamata's Tauberian theorem to these expressions, one finds the following asymptotic behavior for the correlation functions:

$$C(t) \sim \begin{cases} \dfrac{\nu}{\nu^2-1}\Gamma(2/\nu+1)t^{-2/\nu} & \text{for } 1<\nu<\infty \\ \dfrac{1}{1-\nu^2}\Gamma(1/\nu+2)t^{-1-1/\nu} & \text{for } 0<\nu<1. \end{cases} \tag{23}$$

For deriving these laws we had to exclude the integer values of β corresponding to $\nu=2,1,\frac{1}{2},\frac{1}{3},\ldots$. For integer β the hypergeometric functions are degenerate and Eq. (19) is no longer valid. Instead one uses that $F(1,m+1;m+2;z)$ for $m=0,1,2,\ldots$ obeys the relation [6] $F(1,m+1;m+2;z) =\frac{(m+1)(-1)^m}{m!}\frac{d^m}{dz^m}[(1-z)^m F(1,1;2;z)]$, with $F(1,1;2;z) =-\frac{1}{z}\ln(1-z)$. From this one can derive by induction that

$$F(1,m+1;m+2;z) = -(m+1)\left[z^{-(m+1)}\ln(1-z)+\sum_{l=1}^{m}\frac{1}{m+1-l}z^{-l}\right]. \tag{24}$$

So, for integer $\beta=n+1$ Eq. (19) is replaced by

$$\frac{1}{z-1}F\left(1,n+1;n+2;\frac{1}{1-z}\right) = (n+1)\left[(1-z)^n\ln\left(\frac{z}{z-1}\right)+\sum_{l=0}^{n-1}\frac{1}{n-l}(1-z)^l\right]. \tag{25}$$

Here we see that the nth derivative becomes singular and diverges asymptotically as $(-1)^{n+1}\Gamma(n+2)\ln(z-1)$ for $z\to 1$. Correspondingly we find

$$\nu=2,\quad \beta_1=1:\quad \tilde{C}(z)\sim\tilde{C}_1(z)\sim -\frac{2}{3}\ln(z-1),$$

$$\nu=\frac{1}{n},\quad \beta_2=1+n=3,4,\ldots:$$

$$\tilde{C}^{(n)}(z)\sim\tilde{C}_2^{(n)}(z)\sim\frac{1}{1-\nu^2}(-1)^{n+1}\Gamma(n+2)\ln(z-1). \tag{26}$$

Using Karamata's theorem for these expressions we find

$$\nu=2,\quad \beta_1=1:\quad C(t)\sim\frac{2}{3}t^{-1}, \tag{27}$$

$$\nu=\frac{1}{n},\quad \beta_2=1+n=3,4,\ldots:\quad C(t)\sim\frac{1}{1-\nu^2}\Gamma(n+2)t^{-n-1}, \tag{28}$$

which is seen to coincide with the law Eq. (23) at the previously excluded values. Therefore the correlation decay as given by Eq. (23) is correct for all ν values in the indicated ranges. Only the crossover value $\nu=1$ corresponding to $\beta_1 =\beta_2=2$ had to be excluded from the beginning since there our starting point, Eq. (17), is not valid. But the case $\nu=1$ has been treated already separately above leading to $C(t) \sim t^{-2}\ln t$, Eq. (14). Note that by summing up both contributions in Eq. (23) and extrapolating in an appropriate way to $\nu=1$ would yield by the cancellation of two diverging terms with opposite sign, a behavior as $C(t)\sim t^{-2}$ and thus would miss just the logarithmic correction present in Eq. (14). Equations (14) and (23) constitute one of the main results of this paper. It says that algebraically decaying correlations are a quite common feature of the output of hysteretic systems as described by the Preisach model with uncorrelated input. Before we consider its consequences in the frequency domain, we give some more explicit results which can be stated for special values of the parameter ν. There exist infinitely many parameter values ν, where $\tilde{C}(z)$, the Z transform of the output correlation function, can be expressed in terms of elementary functions. To see this, note that if ν takes the values $\nu=\frac{2}{k}$, $k\in\mathbb{N}\setminus\{2\}$ we get for $\tilde{C}(z)$ the expression

$$\tilde{C}(z)=\frac{z}{z-1}\frac{k^2}{4-k^2}\left[\frac{2}{k}F\left(1,k;k+1;\frac{1}{1-z}\right) - F\left(1,1+\frac{k}{2};2+\frac{k}{2};\frac{1}{1-z}\right)\right]. \tag{29}$$

The hypergeometric functions $F(1,\beta;\beta+1;z)$ for integer or half-integer values of β appearing in Eq. (29) reduce to expressions involving only elementary functions. For integer values $\beta=n$ the corresponding formula was given above in Eq. (24). For half-integer values a similar formula can be proved by induction using the properties of the hypergeometric function under differentiation [Eq. (15.2.7) in [5]] and the fact that $F(1,\frac{1}{2};\frac{3}{2};z)=z^{-1/2}\operatorname{arctanh}(z^{1/2})$. One finds

$$F\left(1,n+\frac{1}{2};n+\frac{3}{2};z\right) = (2n+1)\left[z^{-(n+1/2)}\operatorname{arctanh}(z^{1/2})-\sum_{l=1}^{n}\frac{1}{2n+1-2l}z^{-l}\right]. \tag{30}$$

As an example we discuss in more detail the simplest case obtained for $k=1$ in Eq. (29). This corresponds to $\nu=2$ and therefore to a triangular input density $\rho(x)=1-|x|$ for $|x|\le 1$ and zero elsewhere. This case is interesting also because for this value $\tilde{C}(z)$ changes it character from being algebraically divergent as in Eq. (20) to the case where the first derivative $\tilde{C}^{(1)}(z)$ diverges algebraically as in Eq. (21), while $\tilde{C}(z)$ itself remains bounded. According to Eq. (26), for $\nu=2$ a logarithmic divergence of $\tilde{C}(z)$ is to be expected. For $k=1$, Eq. (29) reduces to

$$\tilde{C}(z)=-z-\frac{2}{3}z\ln\left(\frac{z-1}{z}\right)+z(z-1)^{1/2}\operatorname{arctanh}[(1-z)^{-1/2}], \tag{31}$$

and we recognize the term responsible for the logarithmic divergence for $z\to 1$. An explicit systematic expansion of $\tilde{C}(z)$ around $z=1$ verifies this behavior:

$$\tilde{C}(z) \sim -\frac{2}{3}\ln(z-1) - \frac{3}{2} + \frac{\pi}{2}(z-1)^{1/2} - \frac{2}{3}[2+\ln(z-1)] \times (z-1) + \frac{\pi}{2}(z-1)^{3/2} + O[(z-1)^2]. \tag{32}$$

The corresponding spectrum $S(\omega)=2\,\mathrm{Re}\{\tilde{C}[\exp(i\omega)]\}-\frac{1}{3}$ expanded around $\omega=0$ is found as

$$S(\omega) = -\frac{4}{3}\ln|\omega| - \frac{7}{3} + \frac{\pi}{\sqrt{2}}|\omega|^{1/2} + \frac{2\pi}{3}|\omega| + O(|\omega|^{3/2}). \tag{33}$$

As expected the logarithmic divergence of $\tilde{C}(z)$ for $z\to 1$ now manifests itself as a logarithmic divergence of the spectrum $S(\omega)$ for $\omega\to 0$.

More generally, the divergent behavior of $\tilde{C}(z)$ or its derivatives $\tilde{C}^{(n)}(z)$ as given in Eqs. (20)–(22) or Eqs. (26) transfers to the spectrum $S(\omega)$ and its derivatives $S^{(n)}(\omega)$ as follows:

$$2<\nu<\infty: \quad S(\omega) \sim \frac{1}{\nu^2-1}\frac{2\pi}{\cos\pi/\nu}|\omega|^{-1+2/\nu}, \tag{34}$$

$$\nu=2: \quad S(\omega) \sim -\frac{4}{3}\ln|\omega|, \tag{35}$$

$$1<\nu<2:$$

$$S^{(1)}(\omega) \sim \frac{1}{\nu^2-1}\frac{2\pi}{\cos\pi/\nu}\left(\frac{2}{\nu}-1\right)|\omega|^{-2+2/\nu}\,\mathrm{sign}(\omega), \tag{36}$$

$$\nu=1: \quad S^{(1)}(\omega) \sim \pi\ln|\omega|\mathrm{sign}(\omega), \tag{37}$$

$$\frac{1}{n}<\nu<\frac{1}{n-1}, \quad n=2,3,\ldots:$$

$$S^{(n)}(\omega) \sim \frac{1}{\nu(\nu-1)}\frac{\pi}{\sin\pi/(2\nu)}\frac{\Gamma(1+1/\nu)}{\Gamma(1+1/\nu-n)}|\omega|^{-n+1/\nu} \times[\mathrm{sign}(\omega)]^n, \tag{38}$$

$$\nu=\frac{1}{n}, \quad n=2,3,\ldots:$$

$$S^{(n)}(\omega) \sim \frac{1}{1-\nu^2}\Gamma(n+2)i^n\ln\frac{1}{1-\exp i\omega} + \mathrm{c.c.} \tag{39}$$

The last equation implies that for $\nu=\frac{1}{n}$ with even integer n the nth derivative $S^{(n)}(\omega)$ of the spectrum diverges logarithmically, $S^{(n)}(\omega\to 0)\sim \pm\frac{1}{1-\nu^2}\Gamma(n+2)\ln(2-2\cos\omega)$, whereas for odd n it exhibits a discontinuity $S^{(n)}(\omega\to 0) \sim \pm\frac{1}{1-\nu^2}\Gamma(n+2)[\omega-\pi\,\mathrm{sign}(\omega)]$. One should emphasize that the result of Eq. (34) means that in the whole range of input densities characterized by an exponent $2<\nu<\infty$, the output of the Preisach transducer exhibits a divergence of the spectrum as $\omega^{-1+\eta}$ with $0<\eta<1$. We thus found a new mechanism for the generation of $1/f$ noise. We note that the superposition of infinitely many exponentially decaying contributions leading to the long-time tails and to $1/f$ noise is formally similar to the superposition of Lorentzians as discussed, e.g., in Ref. [8]. The involved physical processes, however, are quite different and more complicated. In Fig. 1 examples for the obtained behavior of the spectral density are shown for selected values of ν. The behavior of the corresponding output correlation functions is provided in Fig. 2.

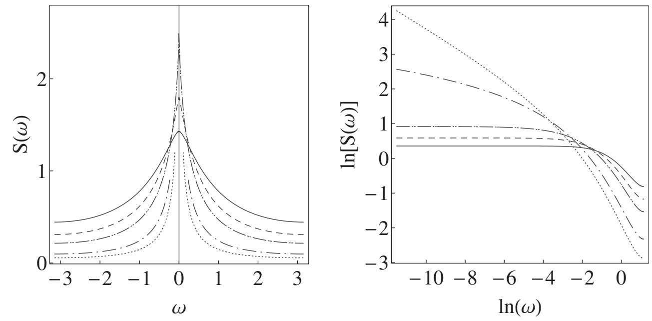

FIG. 1. The exact form of the output spectral density according to Eq. (11) (for $\nu=1$) and Eq. (17) (for $\nu\neq 1$) is shown for several input distributions with charcteristic exponents ν (left: linear; right: double-logarithmic scale). For $\nu=3$ (dotted) the spectrum $S(\omega)$ diverges like $|\omega|^{-1/3}$ and for $\nu=2$ (dot-dashed) logarithmically. For $\nu=1$ (dot-dot-dashed) the first derivative $S^{(1)}(\omega)$ diverges logarithmically, for $\nu=2/3$ (dashed) $S^{(2)}(\omega)$, and for $\nu=2/5$ (full) $S^{(3)}(\omega)$ diverges like $|\omega|^{-1/2}$.

The degree of nonanalyticity of $\tilde{C}(z)$ or the spectrum $S(\omega)$, as reflected in Eqs. (20)–(22) and Eqs. (34)–(39) can be understood rather easily, without doing the detailed asymptotic expansions. The degree of nonanalyticity of a function $f(x)$ at x_0 is defined as the smallest integer n, for which $f^{(n)}(x_0)$, the nth derivative of f at x_0 is discontinuous. The above results imply for the degree of nonanalyticity n of $\tilde{C}(z)$ at $z=1$, or, equivalently, of the spectral density $S(\omega)$ at $\omega=0$,

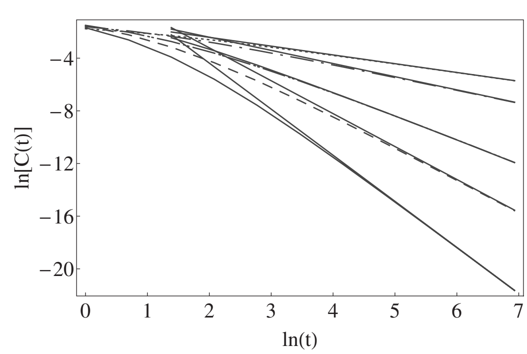

FIG. 2. The output correlation functions as obtained by numerical Fourier transformation of the spectra in Fig. 1 (same plot style) are shown on a double-logarithmic scale. For comparison their asymptotic behavior according to Eqs. (14) and (23) is also shown (full lines). From top to bottom they decay as $t^{-2/3}$, t^{-1}, $t^{-2}\ln t$, $t^{-5/2}$, and $t^{-7/2}$.

$$n = \lceil \min(\gamma, 2\gamma - 1) \rceil, \tag{40}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Here we introduced, also for later generalizations, the exponent $\gamma = 1/\nu$. To obtain this result without complicated calculations it is sufficient to discuss the integrals for $\tilde{C}(z=1)$ in Eq. (9) and its derivatives $\tilde{C}^{(n)}(z=1)$. Inserting in accordance with Eq. (16) the effective Preisach density

$$\tilde{\mu}(u) = \gamma u^{-1+\gamma}, \quad \gamma > 0, \tag{41}$$

and the identity $\frac{d^n}{dz^n}\frac{z}{z-1+v}\big|_{z=1} = n!(-1)^n(v^{-n-1} - v^{-n})$, one obtains for $\tilde{C}^{(n)}(z=1)$ after transforming to polar coordinates $(u,v) = (r \sin \varphi, r \cos \varphi)$ the expression $\tilde{C}^{(n)}(z=1) = \gamma^2 n!(-1)^n(F_n - F_{n-1})$, with

$$F_n = \int_{\pi/4}^{\pi/2} d\varphi (\sin \varphi)^{\gamma-2}(\cos \varphi)^{\gamma-n-1}\int_0^{1/\sin \varphi} dr\, r^{2\gamma-n-2} + \int_0^{\pi/4} d\varphi (\sin \varphi)^{\gamma}(\cos \varphi)^{\gamma-n-3}\int_0^{1/\cos \varphi} dr\, r^{2\gamma-n-2}. \tag{42}$$

The integrals in F_n are seen to diverge due to the cos φ term at $\varphi = \pi/2$ for $n \geq \gamma$ and due to the radial component at $r = 0$ for $n \geq 2\gamma - 1$ and analogously in F_{n-1}. Since the smallest integer n determines the degree of nonanalyticity, one immediately gets from these inequalities the result of Eq. (40).

D. Power-law Preisach and input density

We now generalize the above results to the case where on a finite support, both the input density $\rho(x)$ and the Preisach density $\mu(\alpha)$ take the form of a power law. We assume as in the previous paragraph that the density of the uncorrelated input signal is given by $\rho(x) = \frac{\nu}{2}(1 - |x|)^{\nu-1}$ for $|x| \leq 1$, and zero elsewhere, with $\nu > 0$. In addition, we now generalize the form of the Preisach density $\mu(\alpha)$, the distribution of thresholds, to $\mu(\alpha) = \nu'(1-\alpha)^{\nu'-1}$ for $0 \leq \alpha \leq 1$, and zero elsewhere, $\nu' > 0$. The previous results are seen to correspond to the special case $\nu' = 1$. Inserting $\alpha(u)$, the inverse function of $u(\alpha) = 2F(\alpha, \infty)$, which as before is given by $\alpha(u) = 1 - u^{1/\nu}$ into the formula for the effective Preisach density, Eq. (10), one obtains for the latter exactly the form Eq. (41) $\tilde{\mu}(u) = \gamma u^{-1+\gamma}$, with $\gamma = \frac{\nu'}{\nu}$. This implies that all results for constant Preisach density of the previous section, such as the power-law decay of the output correlations or the appearance of $1/f$ noise, apply to the case with distinct power-law densities for the input and Preisach density, respectively, under the simple replacement $\nu \to \nu/\nu'$.

E. Algebraically and exponentially decaying densities

It turns out that also, cases where the input density $\rho(x)$ and the Preisach density $\mu(\alpha)$ have an infinite support and decay algebraically or exponentially, can be treated exactly. In the first case, we assume for the input density the form $\rho(x) = \frac{\nu}{2}(1+|x|)^{-\nu-1}$, with $\nu > 0$, and for the Preisach density we take the same form, but with parameter ν', i.e., $\mu(\alpha) = \nu'(1+\alpha)^{-\nu'-1}$ for $\alpha > 0$. For this input density $\rho(x)$ one obtains for $u(\alpha) = 2F(\alpha, \infty) = 2\int_\alpha^\infty \rho(x)dx$, the result $u(\alpha) = (1+\alpha)^{-\nu}$ with inverse function $\alpha(u) = u^{-1/\nu} - 1$. Inserting the latter into Eq. (10) for the effective Preisach density $\tilde{\mu}(u)$, one obtains for $\tilde{\mu}(u)$ again, exactly the form of Eq. (41) with $\gamma = \frac{\nu'}{\nu}$. Therefore all results of Sec. III C are valid also for algebraically decaying densities with characteristic exponents ν and ν', respectively, if in the formulas of Sec. III C, ν is replaced by ν/ν'.

Interestingly, the same result is obtained also for exponentially decaying densities: Assuming for the Preisach density the form $\mu(\alpha) = \nu' \exp(-\nu'\alpha)$ for $\alpha > 0$, and for the input density the functional form $\rho(x) = \frac{\nu}{2}\exp(-\nu|x|)$, with ν' and $\nu > 0$, we get from the latter $u(\alpha) = \nu \exp(-\nu\alpha)$, and therefore $\alpha(u) = -1/\nu \ln(u)$. Inserting this into Eq. (10) results again in the effective Preisach density $\tilde{\mu}(u) = \frac{\nu'}{\nu}u^{-1+\nu'/\nu}$, and therefore all results of Sec. III C can also be transferred to exponentially decaying densities.

It is of considerable interest not only to consider cases where both densities decay in the same way, but also cases where one density decays exponentially and the other algebraically. Let us first treat an algebraically decaying input density, i.e., $\rho(x) = \frac{\nu}{2}(1+|x|)^{-\nu-1}$, and an exponentially decaying Preisach density $\mu(\alpha) = \nu' \exp(-\nu'\alpha)$. For this combination one obtains for the effective Preisach density

$$\tilde{\mu}(u) = \nu' \exp(\nu')u^{-1-1/\nu}\exp(-\nu' u^{-1/\nu}). \tag{43}$$

For this density we cannot calculate the integrals for $\tilde{C}(z)$ in Eq. (9) analytically. We can, however, say something about the degree of nonanalyticity of $\tilde{C}(z)$. Since all derivatives of $\tilde{\mu}(u)$ at $u=0$ vanish, this effective density corresponds, roughly speaking, to the case $\gamma \to \infty$ in Eq. (41). Therefore all derivatives $\tilde{C}^{(n)}(z=1)$ exist and the degree of nonanalyticity of $\tilde{C}(z=1)$ or the spectrum $S(\omega=0)$ is infinite. This means that $\tilde{C}(z)$ and $S(\omega)$ are analytic, and therefore the output correlation function does not exhibit long-time tails.

The other combination, where the Preisach density $\mu(\alpha)$ is the broad distribution, i.e., algebraically decaying with $\mu(\alpha) = \nu'(1+\alpha)^{-\nu'-1}$, and the input density $\rho(x)$ is narrow, i.e., exponentially decaying with $\rho(x) = \frac{\nu}{2}\exp(-\nu|x|)$, gives for the effective Preisach density

$$\tilde{\mu}(u) = \frac{\nu'}{\nu}\left(1 - \frac{1}{\nu}\ln u\right)^{-\nu'-1} u^{-1}, \tag{44}$$

which corresponds apart from logarithmic corrections to the limit $\gamma \to 0$ in the effective Preisach density, Eq. (41). This implies that in this case already $\tilde{C}(z=1)$ or the spectrum $S(\omega=0)$ is divergent, i.e., the degree of nonanalyticity is $n=0$. In this sense this combination of input and Preisach density leads to $1/f$ noise in the output signal.

F. Miscellaneous results

In view of the above equivalence results one wonders whether there are other functional forms of the input and the Preisach density leading to the same effective Preisach density, e.g., that of Eq. (41), and the corresponding dynamical behavior. There are actually infinitely many pairs $\{\rho(x),\mu(\alpha)\}$ leading to the same $\tilde{\mu}(u)$. This may be seen already from Eq. (10), but becomes clearer by rewriting this equation as the transformation law of the density $\mu(\alpha)$ under the monotonously decreasing map $u(\alpha)$,

$$\tilde{\mu}(u) \equiv \int_0^\infty \mu(\alpha)\delta(u-u(\alpha))d\alpha, \tag{45}$$

with $u(\alpha)=2\int_\alpha^\infty \rho(x)dx$, or in its local form $\tilde{\mu}(u)du = -\mu(\alpha)d\alpha$. The equivalence of Eqs. (10) and (45) follows from the properties of the Dirac delta distribution with its argument being a function. So for instance, for any form of $\rho(x)$ one can determine the Preisach density $\mu(\alpha)$, which leads to a given $\tilde{\mu}(u)$ by

$$\mu(\alpha) = \tilde{\mu}(u(\alpha))|u'(\alpha)| = 2\tilde{\mu}(u(\alpha))\rho(\alpha). \tag{46}$$

As an example we consider the case, important in practice, of Gaussian input densities $\rho(x)=(2\pi\sigma^2)^{-1/2}\exp[-x^2/(2\sigma^2)]$, and ask which kind of Preisach density $\mu(\alpha)$ leads to the simple form $\tilde{\mu}(u)=\gamma u^{-1+\gamma}$ of the effective Preisach density as in Eq. (41), so that the results of Sec. III C apply. For Gaussian input $\rho(x)$ the function $u(\alpha)$ is given by $u(\alpha) = \mathrm{erfc}(\alpha/\sqrt{2\sigma^2})$, where $\mathrm{erfc}(x)$ is the complementary error function $\mathrm{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty \exp[-t^2]dt$. Inserting this expression for $u(\alpha)$ into Eq. (46), one finds for the Preisach density the exact form

$$\mu(\alpha) = 2\gamma[\mathrm{erfc}(\alpha/\sqrt{2\sigma^2})]^{-1+\gamma}(2\pi\sigma^2)^{-1/2}\exp[-\alpha^2/(2\sigma^2)]. \tag{47}$$

We have seen at the end of Sec. III B that the behavior of $\rho(x)$, respectively $u(\alpha)$, for large arguments determines the long-time behavior of the output correlations. Via our transformation Eq. (10), or Eq. (45), this enters the formula Eq. (9) through the near-zero u behavior of $\tilde{\mu}(u)$. The latter is in addition affected by the large α behavior of the Preisach density $\mu(\alpha)$. Therefore it is of interest to characterize the large α behavior of $\mu(\alpha)$ from Eq. (47). With the asymptotic expansion of the complementary error function $\mathrm{erfc}(x)\sim(\sqrt{\pi}x)^{-1}\exp(-x^2)$ [5], one gets $\mu(\alpha)\sim\gamma/\sigma^2(2\sigma^2/\pi)^{\gamma/2}\alpha^{1-\gamma}\exp[-\gamma\alpha^2/(2\sigma^2)]$. Therefore the tail of $\mu(\alpha)$ is asymptotically given by $\ln\mu(\alpha)\sim-\gamma\alpha^2/(2\sigma^2)$. So apart from logarithmic corrections the large α behavior of $\mu(\alpha)$ from Eq. (47) is Gaussian with variance $\sigma'^2=\sigma^2/\gamma$. Conversely, for given Gaussian input and Preisach density with variance σ^2 and σ'^2, respectively, one finds an effective Preisach density $\tilde{\mu}(u)$, which apart from logarithmic corrections behaves for small u as $u^{-1+\sigma^2/\sigma'^2}$. This means that the results of Sec. III C, such as the power-law decay of Eq. (23), or the $|\omega|^{-1+2/\nu}$-spectral divergence of Eq. (34), hold also in this case if the exponent ν is identified with $\nu = \sigma'^2/\sigma^2$. The results for Gaussian input and Preisach densities, e.g., on the power-law decay of the output correlations, were confirmed also by direct numerical integration of Eq. (9) in the time domain. These numerical results, however, do not detect logarithmic corrections to these laws, which may be present due to the above arguments.

Here one may also wonder what is obtained if exponentially decaying and Gaussian densities are combined for the input and Preisach density, respectively. Again, only the degree of nonanalyticity can be calculated: Taking the "broad" distribution, the exponential distribution, for the input density and the "narrow" one, the Gaussian, for the Preisach density, one finds, similar to Sec. IIIE, that the degree of nonanalyticity is infinite and no long-time tails occur. In the reversed case, narrow input and broad Preisach density, the other extreme degree of nonanalyticity $n=0$ and correspondingly $1/f$ noise in the output signal is obtained. Of course, the same scenario holds for the combination of algebraically decaying (broad) and Gaussian (narrow) densities.

Finally, let us briefly consider the extreme case of a narrow distribution for the Preisach density, a distribution with a finite support strictly contained in the support of the input density. We denote the maximal α value for which the Preisach density is nonzero as α_{max}, which is assumed to be strictly smaller than x_{max}, the maximum of the input signal. This implies according to Eq. (45) that there exists a nonzero value $u_{\mathrm{min}}=u(\alpha_{\mathrm{max}})=2F(\alpha_{\mathrm{max}},\infty)$ below which the effective Preisach density $\tilde{\mu}(u)$ is exactly zero with the consequence that according to Eq. (9) with the lower integration limits u_{min} instead of zero, again all derivatives $\tilde{C}^{(n)}(z=1)$ are finite and no long-time tails occur. Since correspondingly the possible relaxation rates $\lambda=|\ln[1-2F(\alpha,\infty)]|$ in Eq. (8) have a lower bound $\lambda_{\mathrm{min}}=|\ln[1-2F(\alpha_{\mathrm{max}},\infty)]|$ strictly larger than zero, one obtains asymptotically an exponential decay of the output correlations. Depending on the value of $F(\alpha_{\mathrm{max}},\infty)$, possibly a crossover from power-law decay to exponential decay can be observed. The reversed situation, input support contained within the support of the Preisach density, simply corresponds to a rescaling of the Preisach density, with the consequence that all of the above results on long-time tails hold also in this case.

G. Universality classes

In the previous sections we gave explicit examples of function classes for the input and Preisach densities for which we could prove that they lead to long-time tails in the autocorrelation function of the output. Here we discuss briefly the inverse problem and ask: Given an asymptotic decay of the output correlation function of the form

$$C(t) \sim ct^{-\eta}, \quad 0<\eta<\infty, \tag{48}$$

what are the systems leading to this behavior with a given exponent η? Or, more specifically and in view of the fact that for $0<\eta<1$ Eq. (48) is equivalent to a low-frequency behavior of $S(\omega)$ of the form

$$S(\omega) \sim a\omega^{-1+\eta}, \tag{49}$$

what are the systems resulting in $1/f$ noise? The answer is simply obtained by reverting the arguments leading from the form of the effective density $\tilde{\mu}(u)$ to the result Eq. (23). One finds that for a prescribed exponent η the effective density $\tilde{\mu}(u)$ must have a small-u behavior of the form

$$\tilde{\mu}(u) \sim cu^{-1+\gamma} \text{ with } \gamma = \begin{cases} \eta/2 & \text{for } 0 < \eta < 2 \\ \eta - 1 & \text{for } 2 < \eta < \infty, \end{cases} \quad (50)$$

where $c=c(\eta)$ is a constant independent of u, or only weakly, i.e., logarithmically dependent on u, if logarithmic corrections of the long-time tails are allowed for. It is only the small-u behavior which matters, because according to Eq. (9) this leads to the divergence of the decay times. Correspondingly, due to Eq. (45) it is the functional dependence of the input density $\rho(x)$ and the Preisach density $\mu(\alpha)$ for large arguments, which determines γ or η in Eq. (50). Requiring a behavior $\tilde{\mu}(u) \sim cu^{-1+\gamma}$ for $u \to 0$ means, according to Eq. (46) that μ and ρ for large arguments must obey $\mu(\alpha) \sim 2c\rho(\alpha)[\int_\alpha^\infty \rho(x)dx]^{\gamma-1}$ or its differential form

$$\frac{d}{d\alpha}[\mu(\alpha)/\rho(\alpha)]^{1/\gamma-1} \sim c_\gamma \rho(\alpha), \quad (51)$$

with c_γ a negative constant simply related to c. The latter differential equation has to be solved for ρ if one wants to determine for a given Preisach density μ the input density leading to a desired exponent γ or via Eq. (50) to a long-time tail with exponent η. Of course, all the examples of Secs. III C–III E fulfill Eq. (51) exactly with $\gamma=\nu'/\nu$, i.e., as an equality, while in general only asymptotic equality is required.

IV. CONCLUSION AND DISCUSSION

We have shown analytically that the Preisach model, the most prominent and simplest model for complex hysteresis [9], is under very general circumstances able to transform uncorrelated input into output with long-time correlations. The form of the long-time tails depends solely on the tails of the input and Preisach density. Specifically, for symmetric Preisach models we were able to determine the exact asymptotic behavior of the output correlations or low-frequency spectral densities, if both the input density and the Preisach density belong to the same class of functions. Examples are exponentially or algebraically decaying densities. The only requirement for the occurrence of long-time tails is that an associated effective Preisach density behaves for small arguments u algebraically as $u^{-1+\gamma}$ with $0<\gamma<\infty$. This behavior also determines the universality classes of input and Preisach density combinations leading to the same long-time tail. For small enough γ the spectral density even diverges leading to $1/f$ noise in the output signal. The mechanism is formally similar to the superposition of Lorentzians discussed in Ref. [8]. The difference lies in the more complicated form of the weighting functions, which in addition are nontrivially related to the properties of the input and the distribution of elementary hysteresis relays. If the Preisach and input density belong to different function classes one obtains $1/f$ noise, if the Preisach density belongs to the class containing the "broader" functions, e.g., algebraically decaying functions, while the input density is decaying exponentially. In the reverse case, broad input and narrow Preisach density, no long-time tails, but typically exponentially decaying output correlations are observed.

Our results were derived for input series in discrete time with time differences $\Delta t=1$. Since long-time tails have no characteristic scale, the calculated exponents are independent of the value of Δt. Especially, the results hold for arbitrarily small Δt, and therefore also in the white noise limit in continuous time. Obviously, if the time differences of subsequent input events vary stochastically around some mean value, the average decay of the long-time tails will not be affected. The role of correlations in the input time series has not yet been explored analytically. However, numerical results for input according to the Ornstein-Uhlenbeck process indicate that also in this case long-time tails are induced by the hysteretic system [10]. Although the basic mechanisms for long-time tails found here for symmetric Preisach models apply in principle also for general, nonsymmetric Preisach models, the situation there is more complicated due to the additional degrees of freedom in the Preisach density. Especially the question of universality classes with respect to the type of correlation decay will be explored in future publications. Finally, it will be of interest to see whether our approach can be extended to explain analytically recent results [11] on the occurrence of stochastic resonance in systems modeled by the Preisach nonlinearity.

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